Differential Geometry III: Riemannian geometry

4.1 Let $\mathcal{M}$ be a differentiable manifold and ∇ a connection on $\mathcal{M}$.

(a) Show that there exists no $(1,2)$-type tensor field A on $\mathcal{M}$ with the property that, in any local coordinate system $(x^1, \dots, x^n)$ on $\mathcal{M}$

$$A^k_{ij} = \Gamma^k_{ij}.$$

Hint: Check how Γ^k_{ij} transforms under changes of coordinates.

(b) Show that the torsion $T : \Gamma(\mathcal{M}) \times \Gamma(\mathcal{M}) \to \Gamma(\mathcal{M})$ of the connection ∇, which is defined by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$

is a tensor field.

(c) Let $\bar{\nabla}$ be a (possibly) different connection on $\mathcal{M}$. Show that the difference $\nabla - \bar{\nabla} : \Gamma(\mathcal{M}) \times \Gamma(\mathcal{M}) \to \Gamma(\mathcal{M})$ is also a tensor field. Deduce that, there exists a $(1,2)$-type tensor field A such that, in any given local coordinate system $(x^1, \dots, x^n)$,

$$A^k_{ij} = \Gamma^k_{ij} - \bar{\Gamma}^k_{ij}$$

where Γ^k_{ij} and $\bar{\Gamma}^k_{ij}$ are the Christoffel symbols of ∇ and $\bar{\nabla}$, respectively.

(d) Show that, if $h_1, h_2 \in C^\infty(\mathcal{M})$, then $h_1\nabla + h_2\bar{\nabla}$ is again a connection if and only if $h_1 + h_2 = 1$.

Solution. (a) Assume that there exists a tensor field A as in the statement. Then, if $(x^1, \dots, x^n)$ and $(y^1, \dots, y^n)$ are two coordinate systems around the same point $p \in \mathcal{M}$, the components A^k_{ij} and $\tilde{A}^k_{ij}$ of A in the two coordinate systems, respectively, are related by the transformation formula

$$\tilde{A}^k_{ij} = A^\gamma_{\alpha\beta} \cdot \frac{\partial y^k}{\partial x^\alpha} \frac{\partial x^\beta}{\partial y^i} \frac{\partial x^\beta}{\partial y^j}. \tag{1}$$

On the other hand, the Christoffel symbols Γ^k_{ij} and $\tilde{\Gamma}^k_{ij}$ in the coordinate systems $(x^1, \dots, x^n)$ and $(y^1, \dots, y^n)$, respectively, are given by the relations

$$\Gamma^k_{ij} = dx^k\Big(\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}\Big)$$

and

$$\begin{aligned}
\tilde{\Gamma}^k_{ij} &= dy^k\Big(\nabla_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j}\Big) \\
&= \frac{\partial y^k}{\partial x^\gamma} dx^\gamma\Big(\nabla_{\frac{\partial x^\alpha}{\partial y^i}\cdot\frac{\partial}{\partial x^\alpha}} \big(\frac{\partial x^\beta}{\partial y^j} \cdot \frac{\partial}{\partial x^\beta}\Big) \\
&= \frac{\partial y^k}{\partial x^\gamma} \cdot \frac{\partial x^\alpha}{\partial y^i} \cdot dx^\gamma\Big(\nabla_{\frac{\partial}{\partial x^\alpha}} \big(\frac{\partial x^\beta}{\partial y^j} \cdot \frac{\partial}{\partial x^\beta}\Big)
\end{aligned}$$

$$\begin{aligned}
&= \frac{\partial y^k}{\partial x^\gamma}\cdot\frac{\partial x^\alpha}{\partial y^i}\cdot dx^\gamma\Big(\frac{\partial}{\partial x^\alpha}\big(\frac{\partial x^\beta}{\partial y^j}\big)\cdot\frac{\partial}{\partial x^\beta}+\frac{\partial x^\beta}{\partial y^j}\cdot\nabla_{\frac{\partial}{\partial x^\alpha}}\big(\frac{\partial}{\partial x^\beta}\big)\Big)\\
&= \frac{\partial y^k}{\partial x^\gamma}\cdot\frac{\partial x^\alpha}{\partial y^i}\cdot\Big(\frac{\partial}{\partial x^\alpha}\Big(\frac{\partial x^\beta}{\partial y^j}\Big)\cdot dx^\gamma\big(\frac{\partial}{\partial x^\beta}\big)+\frac{\partial x^\beta}{\partial y^j}\cdot dx^\gamma\big(\nabla_{\frac{\partial}{\partial x^\alpha}}\big(\frac{\partial}{\partial x^\beta}\big)\big)\Big)\\
&= \frac{\partial y^k}{\partial x^\gamma}\cdot\frac{\partial x^\alpha}{\partial y^i}\cdot\frac{\partial}{\partial x^\alpha}\Big(\frac{\partial x^\gamma}{\partial y^j}\Big)+\Gamma^\gamma_{\alpha\beta}\cdot\frac{\partial y^k}{\partial x^\gamma}\cdot\frac{\partial x^\alpha}{\partial y^i}\cdot\frac{\partial x^\beta}{\partial y^j}
\end{aligned}$$

(note that we used the fact that that $dx^k(\cdot)$ is a tensor field and, thus, is $C^\infty(\mathcal{M})$-linear in its argument). Therefore, we see that the transformation law for the Christoffel symbols contains an additional term which is not there in (1), namely $\frac{\partial y^k}{\partial x^\gamma}\cdot\frac{\partial x^\alpha}{\partial y^i}\cdot\frac{\partial}{\partial x^\alpha}$. Expressing the coordinates $y^i = y^i(x)$ as functions of $(x^1,\dots,x^n)$, this term is equal to

$$[Dy]^k_\gamma\cdot\big([Dy]^{-1}\big)^\alpha_i\cdot\Big(\frac{\partial}{\partial x^\alpha}\big([Dy]^{-1}\big)^\beta_j\Big)$$

where $[DY]^i_\alpha = \frac{\partial y^i}{\partial x^\alpha}$ is the Jacobian matrix for y. In particular, if the second derivatives of the transformation $x\to y(x)$ at $p\in\mathcal{M}$ are *not* all 0, then this term will have a non-zero at p. Therefore, Γ^k_{ij} does not transform under coordinate changes like a tensor field.

(b) In order to show that T is a tensor field, it suffices to show that it is $C^\infty(\mathcal{M})$-linear in its arguments; since T obviously satisfies $T(X_1+X_2,Y)=T(X_1,Y)+T(X_2,Y)$ (because ∇ and $[\cdot,\cdot]$ are $\mathbb{R}$-linear in their arguments) and $T(X,Y)=-T(Y,X)$, it suffices to show that, for any $X,Y\in\Gamma(\mathcal{M})$ and $f\in C^\infty(\mathcal{M})$:

$$T(f\,X,Y) = f\,T(X,Y).$$

Recall that the Lie bracket $[\cdot,\cdot]$ satisfies for any

$$[f\,X,Y] = f[X,Y]-Y(f)\cdot X$$

since, for any $h\in C^\infty(\mathcal{M})$:

$$[f\,X,Y](h) = f\,X\big(Y(h)\big)-Y\big(fX(h)\big) = f\,X\big(Y(h)\big)-Y(f)X(h)-f\,Y\big(X(h)\big) = f\,[X,Y](h)-Y(f)X(h).$$

Using the above observation and the fact that ∇ is $C^\infty(\mathcal{M})$ in its first argument and satisfies the Leibniz rule with respect to its second argument, we can calculate:

$$\begin{aligned}
T(f\,X,Y) &= \nabla_{fX}Y-\nabla_Y(fX)-[fX,Y]\\
&= f\nabla_XY - Y(f)X - f\nabla_YX - f[X,Y]+Y(f)X\\
&= f\cdot\big(\nabla_{fX}Y-\nabla_Y(fX)-[X,Y]\big)\\
&= f\,T(X,Y).
\end{aligned}$$

(c) As before, we have to verify that $\nabla-\bar\nabla$ is $C^\infty(\mathcal{M})$-linear in both its arguments; since, by the definition of a connection, both ∇ and $\bar\nabla$ are $C^\infty(\mathcal{M})$-linear in their first argument and $\mathbb{R}$-linear in their second argument, it remains to prove that, for any $X,Y\in\Gamma(\mathcal{M})$ and $f\in C^\infty(\mathcal{M})$:

$$(\nabla-\bar\nabla)(X,f\,Y) = f\,(\nabla-\bar\nabla)(X,Y).$$

Indeed:

$$\begin{aligned}(\nabla-\bar\nabla)(X,f\,Y) &= \nabla_X(f\,Y)-\bar\nabla_X(f\,Y)\\ &= X(f)Y+f\,\nabla_XY-X(f)Y-f\,\bar\nabla_XY\\ &= f\,\nabla_XY-f\,\bar\nabla_XY\\ &= f\,(\nabla-\bar\nabla)(X,Y).\end{aligned}$$

Therefore, setting $A(X,Y)\doteq(\nabla-\bar\nabla)(X,Y)=\nabla_XY-\bar\nabla_XY$, we have shown that $A:\Gamma(\mathcal{M})\times\Gamma(\mathcal{M})\to\Gamma(\mathcal{M})$ is a $(1,2)$-tensor field; it is easy to verify that, in any local coordinate system $(x^1,\dots,x^n)$, the components A^k_{ij} of A take the form

$$A^k_{ij}=\Gamma^k_{ij}-\bar\Gamma^k_{ij}.$$

(d) Let us define $D:\Gamma(\mathcal{M})\times\Gamma(\mathcal{M})\to\Gamma(\mathcal{M})$ by the relation

$$D(X,Y)\doteq h_1\nabla_XY+h_2\bar\nabla_XY.$$

The function D is a connection if and only if it satisfies:

1. $D(f_1X_1+X_2,Y)=f_1D(X_1+f_2X_2,Y)+D(X_2,Y)$ for all $X_1,X_2,Y\in\Gamma(\mathcal{M})$ and $f_1\in C^\infty(\mathcal{M})$.
2. $D(X,aY_1+Y_2)=D(X,Y_1)+aD(X,Y_2)$ for all $X,Y_1,Y_2\in\Gamma(\mathcal{M})$ and $a\in\mathbb{R}$.
3. $D(X,f\,Y)=X(f)Y+fD(X,Y)$ for all $X,Y\in\Gamma(\mathcal{M})$ and $f\in C^\infty(\mathcal{M})$.

Among the properties above, 1 and 2 can be easily verified using the fact that they are satisfied by both ∇ and $\bar\nabla$. For proprty 3, using the fact that ∇ and $\bar\nabla$ satisfy the Leibniz rule, we obtain:

$$\begin{aligned}D(X,f\,Y) &= h_1\nabla_X(fY)+h_2\bar\nabla_X(fY)\\ &= h_1X(f)Y+f\,h_1\nabla_XY+h_2X(f)Y+f\,h_2\bar\nabla_XY\\ &= (h_1+h_2)X(f)Y+f\,D(X,Y).\end{aligned}$$

Therefore, property 3 is satisfied if and only if $h_1+h_2=1$.

4.2 Let $\mathcal{M}$ be a smooth manifold equipped with a connection ∇. We can extend the connection ∇ to a map $\nabla:\Gamma(M)\times Ten^k_l(\mathcal{M})\to Ten^k_l(\mathcal{M})$ (where $Ten^k_l(\mathcal{M})$ is the space of tensor fields on $\mathcal{M}$ of type (k,l)) by the requirements that

- ∇ satisfies the Leibniz rule with respect to tensor products, i.e. for all $X\in\Gamma(M)$

$$\nabla_X(f\otimes g)=\nabla_Xf\otimes g+f\otimes\nabla_Xg,$$

- ∇ commutes with contractions, i.e.

$$\nabla_X(\mathrm{tr}A)=\mathrm{tr}\big(\nabla_XA\big).$$

Show that, in any local coordinate chart $(x^1, \dots, x^n)$, if Γ^k_{ij} are the Christoffel symbols of ∇ then, for every 1-form ω:

$$(\nabla_{\frac{\partial}{\partial x^i}} \omega)_j = \partial_i \omega_j - \Gamma^k_{ij} \omega_k.$$

Moreover, for any (k,l)-tensor field T:

$$(\nabla_{\frac{\partial}{\partial x^a}} T)^{i_1 \dots i_k}{}_{j_1 \dots j_l} = \partial_a T^{i_1 \dots i_k}{}_{j_1 \dots j_l} + \Gamma^{i_1}_{ab} T^{b i_2 \dots i_k}{}_{j_1 \dots j_l} + \dots + \Gamma^{i_k}_{ab} T^{i_1 \dots i_{k-1} b}{}_{j_1 \dots j_l}$$
$$- \Gamma^b_{a j_1} T^{i_1 i_2 \dots i_k}{}_{b j_2 \dots j_l} - \dots - \Gamma^b_{a j_l} T^{i_1 \dots i_{k-1} i_k}{}_{j_1 \dots j_{l-1} b}.$$

Solution. We will start by observing that, for any 1-form ω and any vector field X on $\mathcal{M}$, the function $\omega(X) \in C^\infty(\mathcal{M})$ can be seen as the contraction $\mathrm{tr}(\omega \otimes X)$ of the $(1,1)$-tensor field $\omega \otimes X$; this can be seen clearly in local coordinates, since

$$(\omega \otimes X)^i_j \doteq \omega_i X^j \quad \text{and} \quad \omega(X) = \omega_k X^k.$$

Therefore, using our assumptions that $\nabla_X (f \otimes h) = \nabla_X f \otimes h + f \otimes \nabla_X h$ and ∇ commutes with contractions, we obtain for any $X, Y \in \Gamma(\mathcal{M})$:

$$\begin{aligned} Y\big(\omega(X)\big) &= Y\big(\mathrm{tr}(\omega \otimes X)\big) \\ &= \mathrm{tr}\big(\nabla_Y (\omega \otimes X)\big) \\ &= \mathrm{tr}\big(\nabla_Y \omega \otimes X + \omega \otimes \nabla_Y X\big) \\ &= \nabla_Y \omega(X) + \omega\big(\nabla_Y X\big). \end{aligned}$$

By rearranging the terms in the above identity, we thus obtain:

$$\nabla_Y \omega(X) = Y\big(\omega(X)\big) - \omega\big(\nabla_Y X\big).$$

In any given local coordinate system $(x^1, \dots, x^n)$ on $\mathcal{M}$, if we apply the above formula for $X = \frac{\partial}{\partial x^j}$ and $Y = \frac{\partial}{\partial x^i}$ we obtain:

$$\begin{aligned} \big(\nabla_{\frac{\partial}{\partial x^i}} \omega\big)_j &= \partial_i(\omega_j) - \big(\nabla_{\partial_i} \partial_j\big)^k \omega_k \\ &= \partial_i(\omega_j) - \Gamma^k_{ij} \omega_k. \end{aligned}$$

In particular, if $\omega = dx^k$ is a coordinate 1-form, then

$$\nabla_{\partial_i}(dx^k) = -\Gamma^k_{ij} dx^j.$$

If T is a tensor field of type (k,l), then it can be expressed in a local coordinate system $(x^1, \dots, x^n)$ as before as a linear combination of the coordinate (k,l)-tensor fields $\frac{\partial}{\partial x^{\gamma_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{\delta_1} \otimes \dots \otimes dx^{\delta_l}$, $\gamma_1, \dots, \gamma_k, \delta_1, \dots, \delta_l \in \{1, \dots, n\}$:

$$T = T^{\gamma_1 \dots \gamma_k}{}_{\delta_1 \dots \delta_l} \frac{\partial}{\partial x^{\gamma_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\gamma_k}} \otimes dx^{\delta_1} \otimes \dots \otimes dx^{\delta_l}. \tag{2}$$

Our assumption on the behaviour of ∇ on tensor products and the fact that ∇ satisfies the Leibniz rule implies that, for any $f \in C^\infty(\mathcal{M})$, any $X \in \Gamma(\mathcal{M})$ and any $(Y_{(1)}, \dots, Y_{(1)}, \omega_{(1)}, \dots, \omega_{(l)}) \in \Gamma(\mathcal{M}) \times \dots \times \Gamma(\mathcal{M}) \times \Gamma^*(\mathcal{M}) \times \dots \times \Gamma^*(\mathcal{M})$, we have

$$\begin{aligned}\nabla_X\big(f\, Y_{(1)} \otimes \dots \otimes Y_{(k)} \otimes \omega_{(1)} \otimes \dots \otimes \omega_{(l)}\big) = {} & X(f) Y_{(1)} \otimes \dots \otimes Y_{(k)} \otimes \omega_{(1)} \otimes \dots \otimes \omega_{(l)} \\ & + f\,(\nabla_X Y_{(1)}) \otimes \dots \otimes Y_{(k)} \otimes \omega_{(1)} \otimes \dots \otimes \omega_{(l)} \\ & + \dots + f\, Y_{(1)} \otimes \dots \otimes (\nabla_X Y_{(k)}) \otimes \omega_{(1)} \otimes \dots \otimes \omega_{(l)} \\ & + f\, Y_{(1)} \otimes \dots \otimes Y_{(k)} \otimes \nabla_X(\omega_{(1)}) \otimes \dots \otimes \omega_{(l)} \\ & + \dots + f\, Y_{(1)} \otimes \dots \otimes Y_{(k)} \otimes \omega_{(1)} \otimes \dots \otimes (\nabla_X \omega_{(l)}).\end{aligned}$$

Therefore, applying this formula for the $\nabla_{\frac{\partial}{\partial x^\alpha}}$ derivative of the expression (2) and using the fact that

$$\nabla_{\partial_\alpha} \frac{\partial}{\partial x^i} = \Gamma^j_{\alpha i} \frac{\partial}{\partial x^j}, \quad \nabla_{\partial_\alpha}(dx^i) = -\Gamma^i_{\alpha j} dx^j$$

(the last formula following from our computation of the expression of ∇ acting on 1-forms), we obtain:

$$\begin{aligned}\nabla_{\partial_\alpha} T = {} & (\partial_\alpha T^{\gamma_1 \dots \gamma_k}{}_{\delta_1 \dots \delta_l}) \frac{\partial}{\partial x^{\gamma_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\gamma_k}} \otimes dx^{\delta_1} \otimes \dots \otimes dx^{\delta_l} \\ & + T^{\gamma_1 \dots \gamma_k}{}_{\delta_1 \dots \delta_l}\, \Gamma^\beta_{\alpha \gamma_1} \frac{\partial}{\partial x^\beta} \otimes \dots \otimes \frac{\partial}{\partial x^{\gamma_k}} \otimes dx^{\delta_1} \otimes \dots \otimes dx^{\delta_l} \\ & + \dots + T^{\gamma_1 \dots \gamma_k}{}_{\delta_1 \dots \delta_l}\, \Gamma^\beta_{\alpha \gamma_k} \frac{\partial}{\partial x^{\gamma_1}} \otimes \dots \otimes \frac{\partial}{\partial x^\beta} \otimes dx^{\delta_1} \otimes \dots \otimes dx^{\delta_l} \\ & - T^{\gamma_1 \dots \gamma_k}{}_{\delta_1 \dots \delta_l}\, \Gamma^{\delta_1}_{\alpha \beta} \frac{\partial}{\partial x^{\gamma_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\gamma_k}} \otimes dx^\beta \otimes \dots \otimes dx^{\delta_l} \\ & - \dots - T^{\gamma_1 \dots \gamma_k}{}_{\delta_1 \dots \delta_l}\, \Gamma^{\delta_l}_{\alpha \beta} \frac{\partial}{\partial x^{\gamma_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\gamma_k}} \otimes dx^{\delta_1} \otimes \dots \otimes dx^\beta.\end{aligned}$$

Therefore, considering the $\frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_l}$ component of the above expression (noticing that, in each summand involving Γ, an index of Γ is contracted with one index of T, and we are free to rename those indices as we please), we obtain

$$\begin{aligned}(\nabla_{\frac{\partial}{\partial x^a}} T)^{i_1 \dots i_k}{}_{j_1 \dots j_l} = {} & \partial_a T^{i_1 \dots i_k}{}_{j_1 \dots j_l} + \Gamma^{i_1}_{ab} T^{b i_2 \dots i_k}{}_{j_1 \dots j_l} + \dots + \Gamma^{i_k}_{ab} T^{i_1 \dots i_{k-1} b}{}_{j_1 \dots j_l} \\ & - \Gamma^b_{a j_1} T^{b i_2 \dots i_k}{}_{b j_2 \dots j_l} - \dots - \Gamma^b_{a j_l} T^{i_1 \dots i_{k-1} b}{}_{j_1 \dots j_{l-1} b}.\end{aligned}$$

4.3 Let $\mathcal{M}^n$ be a differentiable manifold.

(a) Show that, for any $X, Y, Z \in \Gamma(\mathcal{M})$:

$$\mathcal{L}_{[X,Y]} Z = \mathcal{L}_X \mathcal{L}_Y Z - \mathcal{L}_Y \mathcal{L}_X Z.$$

Show that the above relation also holds when Z is replaced by any tensor field f of type (k, l), $k, l \in \mathbb{N}$. *(Hint: Check how $\mathcal{L}_X$ behaves on tensor products of the form $f_1 \otimes f_2$.)*

(b) Let g be a Riemannian metric on $\mathcal{M}$. We will say that a vector field $X \in \Gamma(\mathcal{M})$ is a *Killing field* if it generates a flow of *isometries* for g, i.e. if, for any $p \in \mathcal{M}$, the flow map $\Phi : (-\delta, \delta) \times \mathcal{U} \to \mathcal{M}$ associated to X in a neighborhood $\mathcal{U}$ of p satisfies

$$(\Phi_t)^*(g \circ \Phi_t) = g \quad \text{for all } t \in (-\delta, \delta).$$

Show that

$$\mathcal{L}_X g = 0.$$

Show also that, in any local system of coordinates, the above equation takes the form

$$g_{ik}\partial_j X^k + g_{jk}\partial_i X^k + \partial_k g_{ij} X^k = 0$$

(*Hint: Apply the product rule on the expression* $X(g(Y,Z)) = \mathcal{L}_X(g(Y,Z))$ *for suitably chosen vector fields* Y, Z.)

(c) Show that the space $\mathcal{K}$ of Killing fields on $(\mathcal{M}, g)$ is closed under commutation, i.e. that $[X, Y] \in \mathcal{K}$ if $X, Y \in \mathcal{K}$; thus, $\mathcal{K}$ forms a Lie subalgebra of $\Gamma(\mathcal{M})$.

*(d) We will later prove in class that if there exists a point $p \in \mathcal{M}$ and a local system of coordinates around p such that $X|_p = 0$ and $\partial_i X^j|_p = 0$ for all $i, j = 1, \ldots, n$, then X vanishes everywhere on the connected component of $\mathcal{M}$ containing p. Using this fact, can you show that on a connected Riemannian manifold $(\mathcal{M}, g)$ the dimension of $\mathcal{K}$ is at most $\frac{n(n+1)}{2}$? Can you find a basis for the Killing algebra $\mathcal{K}$ on $(\mathbb{R}^n, g_E)$?

Solution. (a) Using the formula $\mathcal{L}_X Y = [X, Y]$ holding for any $X, Y \in \Gamma(\mathcal{M})$, we can readily calculate that the relation

$$\mathcal{L}_{[X,Y]} Z = \mathcal{L}_X \mathcal{L}_Y Z - \mathcal{L}_Y \mathcal{L}_X Z$$

is equivalent to the statement that

$$\big[[X, Y], Z\big] = \big[X, [Y, Z]\big] - \big[Y, [X, Z]\big]$$

which, after rearranging the terms and using the anti-symmetry of $[\cdot, \cdot]$ in its arguments, is equivalent to

$$\big[[X, Y], Z\big] + \big[[X, Y], Z\big] + \big[[X, Y], Z\big] = 0.$$

The above is just Jacobi's identity (see Exercise 3.3).

Using the fact that the Lie derivative commutes with contractions and satisfies the product rule with respect to tensor products, we compute that, for any 1-form ω and any $X, Y \in \Gamma(\mathcal{M})$ (recalling also that $\omega(Y) = \mathrm{tr}(\omega \otimes Y)$:

$$X\big(\omega(Y)\big) = \mathcal{L}_X\big(\mathrm{tr}(\omega \otimes Y)\big) = \mathcal{L}_X \omega(Y) + \omega\big(\mathcal{L}_X Y\big),$$

so that:

$$(\mathcal{L}_X \omega)(Y) = X\big(\omega(Y)\big) - \omega\big([X, Y]\big)$$

(from now on, we will drop the parentheses and write $\mathcal{L}_X\omega(Y)$ meaning $(\mathcal{L}_X\omega)(Y)$). Therefore, we can readily calculate for any $X, Y, Z \in \Gamma(\mathcal{M})$

$$\begin{aligned}\mathcal{L}_X(\mathcal{L}_Y\omega)(Z) &= X\big(\mathcal{L}_Y\omega(Z)\big) - \mathcal{L}_Y\omega([X,Z]) \\ &= X\big(Y(\omega(Z)) - \omega([Y,Z])\big) - Y\big(\omega([X,Z])\big) + \omega\big([Y,[X,Z]]\big) \\ &= X\big(Y(\omega(Z))\big) - X\big(\omega([Y,Z])\big) - Y\big(\omega([X,Z])\big) + \omega\big([Y,[X,Z]]\big)\end{aligned}$$

and, after switching the roles of X, Y:

$$\mathcal{L}_Y(\mathcal{L}_X\omega)(Z) = Y\big(X(\omega(Z))\big) - Y\big(\omega([X,Z])\big) - X\big(\omega([Y,Z])\big) + \omega\big([X,[Y,Z]]\big).$$

Subtracting the above relations (noting that the second and third term in each right hand side cancel out), we obtain

$$\begin{aligned}\mathcal{L}_X(\mathcal{L}_Y\omega)(Z) - \mathcal{L}_Y(\mathcal{L}_X\omega)(Z) &= [X,Y]\big(\omega(Z)\big) + \omega\big([Y,[X,Z]] - [X,[Y,Z]]\big) \\ &= [X,Y]\big(\omega(Z)\big) - \omega\big([[X,Y],Z]\big) \\ &= \big(\mathcal{L}_{[X,Y]}\omega\big)(Z)\end{aligned}$$

(where, in passing from the second to the third line above, we used Jacobi's identity). Since the above relation is true for any $Z \in \Gamma(\mathcal{M})$, we infer that

$$\mathcal{L}_{[X,Y]}\omega = \mathcal{L}_X\mathcal{L}_Y\omega - \mathcal{L}_Y\mathcal{L}_X\omega.$$

In order to prove that the same relation holds for any tensor field T, i.e.

$$\mathcal{L}_{[X,Y]}T = \mathcal{L}_X\mathcal{L}_YT - \mathcal{L}_Y\mathcal{L}_XT, \tag{3}$$

we can argue inductively on the type of T: If the formula is true for all tensor fields of type (k,l), then (due to linearity) (3) will also be true for all tensor fields of type $(k+1,l)$ if it's true for tensors of the form

$$T = \bar{T} \otimes V,$$

where $\bar{T}$ is of type (k,l) and $V \in \Gamma(\mathcal{M})$ (we get the same statement for tensor fields of type $(k,l+1)$ if we replace V with $\omega \in \Gamma^*(\mathcal{M})$). Using the formula

$$\mathcal{L}_XT = \mathcal{L}_X\bar{T} \otimes V + \bar{T} \otimes \mathcal{L}_XV,$$

verifying (3) using that it is true for $\bar{T}$ and V is a simple algebraic exercise. Similarly when V is replaced with $\omega \in \Gamma^*(\mathcal{M})$, to deduce that (3) is true for tensor fields of type $(k,l+1)$ if it's true for tensor fields of type (k,l).

(b) As in the case of the proof of the formula $\mathcal{L}_XY = [X,Y]$ that we saw in class, one way to prove that $\mathcal{L}_Xg = 0$ is by arguing in a local coordinate system where X is of the form $\frac{\partial}{\partial x^1}$ (this is only possible around points $p \in \mathcal{M}$ where $X|_p \neq 0$; as we saw in class, this is enough to verify the formula in the closure of the support of X, while outside the support of X the operator $\mathcal{L}_X$ is identically 0 when acting on vector fields and one forms and, hence, on any tensor field by induction).

In such a coordinate system, the flow map Φ_t associated to X is simply the coordinate translation map $(x^1, x^2, \ldots, x^n) \to (x^1+t, x^2, \ldots, x^n)$; therefore, the matrix of the differential $D\Phi_t$ takes the simple form

$$[D\Phi_t]^i_j = \frac{\partial \Phi^i_t}{\partial x^j} = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \ldots & 1 \end{bmatrix}.$$

The fact that $(\Phi_t)^*(g \circ \Phi_t) = g$ is equivalent to the statement that, for any $Y, Z \in \Gamma(M)$ and any $p \in \mathcal{M}$:

$$g|_{\Phi_t(p)}\big(d\Phi_t(Y), d\Phi_t(Z)\big) = g|_p(Y, Z).$$

Applying the above relation for Y, Z being coordinate vector fields in the above coordinate chart, we infer that the components of g satisfy for any t small enough:

$$g_{ij}(x^1+t, x^2, \ldots, x^n) = g_{ij}(x^1, x^2, \ldots, x^n)$$

and, therefore:

$$\frac{\partial g_{ij}}{\partial x^1} = 0. \tag{4}$$

The Lie derivative $\mathcal{L}_X T$ of any tensor field T defined on the coordinate chart where $X = \frac{\partial}{\partial x^1}$ takes the simple form:

$$\big(\mathcal{L}_X T\big)^{i_1\ldots i_k}{}_{j_1\ldots j_l} = \big(\mathcal{L}_{\partial_1} T\big)^{i_1\ldots i_k}{}_{j_1\ldots j_l} = \frac{\partial}{\partial x^1}(T^{i_1\ldots i_k}{}_{j_1\ldots j_l}). \tag{5}$$

We have already seen that this formula is true when T is a vector field; using the formula $\mathcal{L}_X\omega(Y) = X\big(\omega(Y)\big) - \omega([X, Y])$ for $Y = \frac{\partial}{\partial x^k}$, it can be also verified in the case when T is an 1-form. The case when T is a general (k, l)-tensor can be established inductively using the product rule for $\mathcal{L}_X$. Alternatively, one can deduce (5) by noting that, in any coordinate system $(x^1, \ldots, x^n)$, $\mathcal{L}_{\partial_i}\frac{\partial}{\partial x^j} = 0$ and $\mathcal{L}_{\partial_i} dx^j = 0$, and, therefore,

$$\mathcal{L}_{\partial_i}\Big(\frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_l}\Big) = 0.$$

The condition (4) now implies the required relation when $T = g$:

$$(\mathcal{L}_X g)_{ij} = 0.$$

In view of the fact that $\mathcal{L}_X$ commutes with contractions and satisfies the product rule $\mathcal{L}_X(f \otimes h) = \mathcal{L}_X f \otimes h + f \otimes \mathcal{L}_X h$, we can calculate for any $X, Y, Z \in \Gamma(\mathcal{M})$:

$$\begin{aligned} X\big(g(Y, Z)\big) &= (\mathcal{L}_X g)(Y, Z) + g(\mathcal{L}_X Y, Z) + g(Y, \mathcal{L}_X Z) \\ &= g(\mathcal{L}_X Y, Z) + g(Y, \mathcal{L}_X Z). \end{aligned}$$

In any local coordinate system $(x^1, \dots, x^n)$, choosing $Y = \partial_i$ and $Z = \partial_j$, the above formula yields (noting that $[X, Y]^k = -\partial_i X^k$ and $[X, Z]^k = -\partial_j X^k$ in this case):

$$\begin{aligned} X^k \partial_k g_{ij} &= g_{kl} \cdot [X, \partial_i]^k \cdot (\partial_j)^l + g_{kl} \cdot (\partial_i)^l \cdot [X, \partial_j]^i \\ &= -g_{ik} \partial_j X^k - g_{jk} \partial_i X^k. \end{aligned}$$

Remark. Note that our argument in a local coordinate system where $X = \frac{\partial}{\partial x^1}$ in fact yields the following coordinate-independent relation for any covariant tensor (i.e. of type $(0, k)$):

$$\mathcal{L}_X T|_p = \lim_{t \to 0} \frac{(\Phi_t)^* T|_{\Phi_t(p)} - T|_p}{t} \tag{6}$$

where $(\Phi_t)^*$ is the pull-back map associated to Φ_t, i.e. $(\Phi_t)^* T(X_1, \dots, x_k) \doteq T\big(D\Phi_t(X_1), \dots D\Phi_t(X_k)\big)$. Returning, now, to the case when X is a Killing field of g and choosing $T = g$ in the above formula, we have

$$(\Phi_t)_* g|_{\Phi_t(p)} = g|_p \quad \text{for all } p \in \mathcal{M}.$$

Therefore, formula (6) directly implies that $\mathcal{L}_X g = 0$.

(c) If $X, Y \in \mathcal{K}$, then $\mathcal{L}_X g = \mathcal{L}_Y g = 0$. Using the commutator formula from part (a) of this exercise, we calculate

$$\mathcal{L}_{[X,Y]} g = \mathcal{L}_X\big(\mathcal{L}_Y g\big) - \mathcal{L}_Y\big(\mathcal{L}_X g\big) = 0 - 0 = 0.$$

Therefore, is will follow that $[X, Y]$ is also a Killing vector field once we show that, for any vector field Z, the condition

$$\mathcal{L}_Z g = 0$$

implies that Z generates a flow Φ_t of isometries. It suffices to prove this fact at points $p \in \text{supp}(Z) = \big\{q \in \mathcal{M} : Z|_q \neq 0\big\}$ (since, by continuity, the statement will be then true also on $\text{clos}\Big(\text{supp}(Z)\Big)$). The statement is trivially true on the set $\mathcal{M} \setminus \text{clos}\Big(\text{supp}(Z)\Big)$ which consists of points q for which $Z = 0$ on a whole open neighborhood of q; at such points, $\Phi_t = \Phi_0 = \text{Id}$, and hence Φ_t is (trivially) an isometry.

For any $p \in \text{supp}(Z)$, let us choose a local coordinate system $(x^1, \dots, x^n)$ such that $Z = \frac{\partial}{\partial x^1}$. In this coordinate system, $\Phi_t(x^1, x^2, \dots, x^n) = (x^1 + t, x^2, \dots, x^n)$. Therefore,

$$\big((\Phi_t)^* g|_{\Phi_t(p)}\big)_{ij} = g_{ij}(x^1(p) + t, x^2(p), \dots, x^n(p)).$$

Moreover, $\mathcal{L}_Z g = 0$ translates to

$$\partial_1 g_{ij} = 0.$$

Integrating the above equation in the x^1 direction, we obtain

$$g_{ij}(x^1(p) + t, x^2(p), \dots, x^n(p)) = g_{ij}(x^1(p), x^2(p), \dots, x^n(p)),$$

i.e. that $\big((\Phi_t)^* g|_{\Phi_t(p)}\big)_{ij} = (g|_p)_{ij}$, as required.

(d) Let $(\mathcal{M}, g)$ be a connected Riemannian manifold. Using the statement that, if $X^k|_p = \partial_i X^j|_p = 0$ for all $i, j, k = 1, \dots, n$ in some local coordinate chart around p and $X \in \mathcal{K}$ then $X = 0$, we deduce (by linearity) that, if $Z_1, Z_2 \in \mathcal{K}$ satisfy $Z_1^k|_p = Z_2^k|_p$ and $\partial_i Z_1^k|_p = \partial_i Z_2^k$ then $Z_1 = Z_2$ on $\mathcal{M}$. This implies that the number of linearly independent Killing vector fields Z on $(\mathcal{M}, g)$ can be at most as many as the components of $Z^k|_p$ and $\partial_i Z^j|_p$. However, the components of $\partial_i Z^j|_p$ are not independent from each other: Using the formula establised in part (b), i.e. that

$$g_{ik}\partial_j X^k + g_{jk}\partial_i X^k + X^k \partial_k g_{ij} = 0,$$

we infer that the matrix $\mathbb{M}_{ij} = g_{ik}\partial_j X^k|_p$ is completely determined once we know $X^k|_p$ and the elements $\mathbb{M}_{ij}$ of $\mathbb{M}$ corresponding to $i > j$ (there are precisely $\frac{n(n-1)}{2}$ such elements $\mathbb{M}_{ij}$). Since the matrix $[g]$ of g is invertible, this implies that, given $X^k|_p$, the matrix $\partial_i X^j|_p$ is restricted to lie in a subspace of dimension $\frac{n(n-1)}{2}$. Therefore, since $X^k|_p$ has n elements, the dimension of the Killing algebra $\mathcal{K}$ cannot exceed $n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$.

In the case of Euclidean space $(\mathbb{R}^n, g_E)$, the vector fields corresponding to translations in the direction of the axes and rotations in the coordinate 2-planes, that is to say the vector fields

$$T_i = \frac{\partial}{\partial x^i}, \quad i = 1, \dots, n\Omega_{ij} = x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i}, \quad i > j \in \{1, \dots, n\}$$

constitute a set of $\frac{n(n+1)}{2}$ linearly independent Killing vector fields. Since $\mathbb{R}^n$ is connected, this is the maximum possible dimension of the Killing algebra $\mathcal{K}_0$ of $(\mathbb{R}^n, g_E)$; as a result, $\{T_i, \Omega_{ij}\}$ constitute a basis for $\mathcal{K}_0$.

4.4 Let X, Y be two smooth vector fields on a 2-dimensional manifold $\mathcal{M}$ such that

$$[X, Y] = 0$$

and let $p \in \mathcal{M}$ such that $X|_p, Y|_p$ are *not* collinear. In this exercise, we will show that there exists a local system of coordinates (y^1, y^2) around p so that $X = \frac{\partial}{\partial y^1}$, $Y = \frac{\partial}{\partial y^2}$.

(a) Show that if $\mathcal{U}$ is a neighborhood of p and $\Phi : (-\delta, \delta) \times \mathcal{U} \to \mathcal{M}$ is the flow map associated to X, then, for any $t \in (-\delta, \delta)$ and $q \in \mathcal{U}$:

$$d\Phi_{-t}\big(Y|_{\Phi_t(q)}\big) = Y|_q.$$

(a) Let $\gamma : I \to \mathcal{M}$ be an integral curve of the vector field Y such that $\gamma(0) = p$. Consider the map $\Psi : \Omega \subset \mathbb{R}^2 \to \mathcal{M}$ defined in a neighborhood Ω of 0 defined by the relation

$$\Psi(t, s) = \Phi_t(\gamma(s)).$$

Show that Ψ is a diffeomorphism on its image when restricted to a small neighborhood of 0. Show also that in the coordinate system associated to the chart Ψ^{-1}:

$$X = \frac{\partial}{\partial x^1}, \quad Y = \frac{\partial}{\partial x^2}.$$

Solution. (a) For any $q \in \mathcal{U}$ as in the statement of the exercise, let us define for $t \in (-\delta, \delta)$ the tangent vector $v_t \in T_q\mathcal{M}$ by

$$v_t \doteq d\Phi_{-t}\big(Y|_{\Phi_t(q)}\big).$$

We need to show that $v_t = Y|_q$ for all $t \in (-\delta, \delta)$. Since $v_0 = Y|_q$, it suffices to show that, for all $t \in (-\delta, \delta)$:

$$\frac{dv_t}{dt} = 0.$$

We can readily calculate:

$$\begin{aligned}
\frac{dv_t}{dt} &= \lim_{s\to 0} \frac{v_{t+s} - v_t}{s} \\
&= \lim_{s\to 0} \frac{d\Phi_{-t-s}\big(Y|_{\Phi_{t+s}(q)}\big) - d\Phi_{-t}\big(Y|_{\Phi_t(q)}\big)}{s} \\
&= \lim_{s\to 0} \frac{d\Phi_{-t}\big(d\Phi_{-s}Y|_{\Phi_s(\Phi_t(q))}\big) - d\Phi_{-t}\big(Y|_{\Phi_t(q)}\big)}{s} \\
&= d\Phi_{-t}\Big(\lim_{s\to 0} \frac{\big(d\Phi_{-s}Y|_{\Phi_s(\Phi_t(q))} - Y|_{\Phi_t(q)}\big)}{s}\Big) \\
&= d\Phi_{-t}\mathcal{L}_X Y|_{\Phi_t(q)},
\end{aligned}$$

where we used the fact that $\Phi_{t_1+t_2} = \Phi_{t_2} \circ \Phi_{t_1}$ for any $t_1, t_2 \in (-\delta, \delta)$ with $t_1 + t_2 \in (-\delta, \delta)$ and, therefore, using the formula for the derivative of the composition of two maps: $d\Phi_{t_1+t_2}|p = d\Phi_{t_2}|_{\Phi_{t_1}(p)} \cdot d\Phi_{t_1}|_p$. Since we assumed that $[X, Y] = 0$ on $\mathcal{M}$, we deduce that

$$\frac{dv_t}{dt} = 0.$$

Therefore, $v_t = Y|_q$ for all $t \in (-\delta, \delta)$.

(b) Let $\Psi : \Omega \subset \mathbb{R} \to \mathcal{M}$ be as defined in the statement of the exercise, i.e..

$$\Psi(t, s) = \Phi_t(\gamma(s))$$

where Φ_t is the flow map of the vector field X and $\gamma(s)$ satisfies $\gamma(0) = p$, $\dot\gamma(s) = Y|_{\gamma(s)}$. Note that $\Psi(0, 0) = p$. In view of the properties of the flow map of a vector field X, we have

$$\partial_t\Psi(t, s) = \partial_t\Phi_t(\gamma(s)) = X|_{\Phi_t(\gamma(s))} = X|_{\Psi(t,s)}. \tag{7}$$

Moreover, since $\Phi_0 = \mathrm{Id}$, we have $\Psi(0, s) = \gamma(s)$ and

$$\partial_s\Psi(0, s) = \partial_s\Phi_0(\gamma(s)) = \dot\gamma(s) = Y|_{\Psi(0,s)}. \tag{8}$$

In order to show that Ψ is a diffeomorphism on its image when restricted to a small neighborhood of the origin, it suffices to show (in view of the inverse function theorem) that the differential map at the origin $D\Psi|_0$ is invertible. Given any local coordinate system (y^1, y^2) around p, we can calculate the matrix $[D\Psi|_0]$ as follows

$$[D\Psi|_0] = \begin{bmatrix} \partial_t\Psi^1(0,0) & \partial_s\Psi^1(0,0) \\ \partial_t\Psi^2(0,0) & \partial_s\Psi^2(0,0) \end{bmatrix} = \begin{bmatrix} X^1|_p & Y^1|_p \\ X^2|_p & Y^2|_p \end{bmatrix},$$

where we used (7) and (8). Since $X|_p$ and $Y|_p$ were assumed to not be collinear, we deduce that $[D\Psi|_0]$ has full rank and is therefore invertible. Thus, there exists an open neighborhood $\mathcal{V} \subset \Omega$ of $(0,0)$ such that $\Psi : \mathcal{V} \to \Psi(\mathcal{V})$ is a diffeomorphism.

In the coordinate chart Ψ^{-1} on $\Psi(\mathcal{V})$, the coordinates (x^1, x^2) are defined as the pull-backs via the map Ψ^{-1} of the Cartesian coordinates on $\mathcal{V}$. This means that if the point $q \in \Psi(\mathcal{V})$ satisfies $q = \Psi(t,s) = \Phi_t(\gamma(s))$, then $(x^1(q), x^2(q)) = (t,s)$. The coordinate vector field $\frac{\partial}{\partial x^1}$ at the point $q \in \Psi(\mathcal{V})$ is simply the tangent vector $\dot{\zeta}|_q$ of the coordinate curve $\zeta^{(1)} = \{x^2 = x^2(q)\}$ parametrized by x^1. Since this curve is simply

$$\zeta^{(1)}(x^1) = \Phi_{x^1}(\gamma(x^2(q))),$$

we infer that

$$\frac{\partial}{\partial x^1}\Big|_q = \dot{\zeta}^{(1)}(x^1(q)) = \partial_t \Phi_{t=x^1(q)}(\gamma(x^2(q))) = X|_q.$$

Similarly, $\frac{\partial}{\partial x^2}|_q$ is the tangent vector to the coordinate curve $\zeta^{(2)} = \{x^1 = x^1(q)\}$ parametrized by x^2, i.e.

$$\zeta^{(2)}(x^2) = \Phi_{x^1(q)}(\gamma(x^2)).$$

Thus,

$$\frac{\partial}{\partial x^2}\Big|_q = \dot{\zeta}^{(2)}(x^2(q)) = \partial_s \Phi_{t=x^1(q)}(\gamma(s = x^2(q))) = d\Phi_{t=x^1(q)}(\dot{\gamma}(s = x^2(q)) = d\Phi_{t=x^1(q)}(Y|_{\gamma(x^2(q))}).$$

Since we proved in part (a) that

$$d\Phi_t(Y|_{\Phi_{-t}(z)}) = Y|_z \quad \text{for all } z \in \mathcal{U},$$

setting $t = x^1(q)$ and $z = \Phi_{x^1(q)}(\gamma(x^2(q))) = \Psi(x^1(q), x^2(q)) = q$, we infer that

$$\frac{\partial}{\partial x^2}\Big|_q = Y|_q.$$

Remark. A faster way to compute the coordinate vector fields is to simply note that, for any local parametrization $\Psi : \mathcal{V} \subset \mathbb{R}^n \to \mathcal{N}^n$ of a manifold N, the coordinate vector fields in the coordinate chart associated to Ψ^{-1} are simply $d\Psi(\frac{\partial}{\partial x^i})$, where $\frac{\partial}{\partial x^i}$ are the coordinate vector fields in $\mathbb{R}^n$.